Lec 39: 11/30/2009

Simultaneous Eigenstates of L^2 and L_z:

As we saw, L^2 and L_z commute and hence have simultaneous eigenstates. Lets show such an eigenstate by $|\alpha\beta\rangle$ where:

$$L^2|\alpha\beta\rangle = \alpha|\alpha\beta\rangle \quad , \quad L_z|\alpha\beta\rangle = \beta|\alpha\beta\rangle$$

Since $[L_z, L_\pm] = \pm\hbar L_\pm$, we find:

$$L_z L_+ |\alpha\beta\rangle = L_+ L_z|\alpha\beta\rangle + \hbar L_+|\alpha\beta\rangle = (\beta+\hbar)L_+|\alpha\beta\rangle$$

$$L_z L_-|\alpha\beta\rangle = L_- L_z|\alpha\beta\rangle - \hbar L_-|\alpha\beta\rangle = (\beta-\hbar)L_-|\alpha\beta\rangle$$

Thus L_+ and L_- change β by $+\hbar$ and $-\hbar$ respectively.

On the other hand, because $[L^2, L_\pm] = 0$, we have:

$$L^2 L_+|\alpha\beta\rangle = \alpha L_+|\alpha\beta\rangle \quad , \quad L^2 L_-|\alpha\beta\rangle = \alpha L_-|\alpha\beta\rangle$$

This implies that:

$$L_+|\alpha\beta\rangle = C_+(\alpha,\beta)\,|\alpha,\beta+\hbar\rangle$$

$$L_-|\alpha\beta\rangle = C_-(\alpha,\beta)\,|\alpha,\beta-\hbar\rangle$$

Wher $C_+(\alpha,\beta)$ and $C_-(\alpha,\beta)$ are some factors.

L_+ increases β upon operating on a given eigenstate, while L_- increases β. However, this cannot go on indefinitely. Note that $L^2 - L_z^2 = L_x^2 + L_y^2$, implying that $L^2 - L_z^2$ is a positive definite operator. Thus its average must be ≥ 0. The average taken in the state $|\alpha\beta\rangle$ is $\alpha - \beta^2$, hence:

$\alpha \geq \beta^2$

Therefore there must be a β_{max} such that:

$L_+|\alpha\beta_{max}\rangle = 0$

In this case:

$L_-L_+|\alpha\beta_{max}\rangle = 0$

$L_-L_+ = L^2 - L_z^2 - \hbar L_z \Rightarrow (L^2 - L_z^2 - \hbar L_z)|\alpha\beta_{max}\rangle = 0$

$\Rightarrow (\alpha^2 - \beta_{max}^2 - \hbar\beta_{max})|\alpha\beta_{max}\rangle = 0 \Rightarrow \boxed{\alpha^2 = \beta_{max}(\beta_{max}+\hbar)}$

Similarly, there must also be a β_{min} such that:

$$L_-|\alpha\beta_{min}\rangle = 0 \Rightarrow L_+L_-|\alpha\beta_{min}\rangle = 0$$

$$L_+L_- = L^2 - L_z^2 + \hbar L_z \Rightarrow (L^2 - L_z^2 + \hbar L_z)\,|\alpha\beta_{min}\rangle = 0$$

$$\Rightarrow (\alpha^2 - \beta_{min}^2 + \hbar\,\beta_{min})\,|\alpha\beta_{min}\rangle = 0 \Rightarrow \boxed{\alpha^2 = \beta_{min}(-\beta_{min} + \hbar)}$$

From these two expressions we find:

$$\boxed{\beta_{max} = -\beta_{min}}$$

Starting at $|\alpha\beta_{max}\rangle$, we can go all the way down to $|\alpha\beta_{min}\rangle$ by operating L_- successively:

$$\begin{tikzcd}
|\alpha\beta_{max}\rangle \arrow[d, shift right, "L_-"'] \\
|\alpha\beta_{max}-\hbar\rangle \arrow[u, shift right, "L_+"'] \arrow[d, shift right, "L_-"'] \\
\vdots \arrow[u, shift right, "L_+"'] \\
|\alpha\beta_{min}+\hbar\rangle \arrow[u, shift right, "L_+"'] \arrow[d, shift right, "L_-"'] \\
|\alpha\beta_{min}\rangle \arrow[u, shift right, "L_+"']
\end{tikzcd}$$

And, vice versa, we can go all the way up to $|\alpha\beta_{max}\rangle$ by successive operation of L_+ starting with $|\alpha\beta_{min}\rangle$.

These lowerings and raisings happen in units of $\hbar$. Since $\beta_{max} = -\beta_{min}$, we find that:

$$\beta_{max} - \beta_{min} = 2\beta_{max} = \hbar j \qquad j = 0, 1, 2, \ldots$$

$$\Rightarrow \boxed{\frac{\beta_{max}}{\hbar} = \frac{j}{2}}$$

For a given j, we have $2j+1$ eigenstates

$|jm\rangle$ where $-j \leq m \leq j$ and:

$$L^2 |jm\rangle = j(j+1)\hbar^2 |jm\rangle$$

$$L_z |jm\rangle = \hbar m |jm\rangle$$

These states form a $2j+1$ dimensional space.

We can also find that (for details, see Shankar):

$$C_+(j,m) = \hbar\sqrt{j(j+1) - m(m+1)} \quad , C_-(j,m) = \hbar\sqrt{j(j+1) - m(m-1)}$$

As expected $C_+(j,j) = 0$ and $C_-(j,-j) = 0$.

For $j=0$ we find a one-dimensional space $|00\rangle$.

For $j=\frac{1}{2}$, we have a two-dimensional space that is spanned by $|\frac{1}{2}\,\frac{1}{2}\rangle$ and $|\frac{1}{2}\,-\frac{1}{2}\rangle$. Denoting them by $\begin{bmatrix}1\\0\end{bmatrix}$ and $\begin{bmatrix}0\\1\end{bmatrix}$ respectively, we find:

$$J_x = \begin{bmatrix} 0 & \frac{\hbar}{2} \\ \frac{\hbar}{2} & 0 \end{bmatrix}, \quad J_y = \begin{bmatrix} 0 & -\frac{i\hbar}{2} \\ \frac{i\hbar}{2} & 0 \end{bmatrix}, \quad J_z = \begin{bmatrix} \frac{\hbar}{2} & 0 \\ 0 & -\frac{\hbar}{2} \end{bmatrix}$$

(we use $\vec{J}$ instead of $\vec{L}$ because of the spin degree of freedom, $\vec{J} = \vec{L} + \vec{S}$)

For $j=1$, we have a three-dimensional space spanned by $|1\,1\rangle$, $|1\,0\rangle$, $|1\,-1\rangle$. In this case we have:

$$J_x = \begin{bmatrix} 0 & \frac{\hbar}{\sqrt{2}} & 0 \\ \frac{\hbar}{\sqrt{2}} & 0 & \frac{\hbar}{\sqrt{2}} \\ 0 & \frac{\hbar}{\sqrt{2}} & 0 \end{bmatrix}, \quad J_y = \begin{bmatrix} 0 & -\frac{i\hbar}{\sqrt{2}} & 0 \\ \frac{i\hbar}{\sqrt{2}} & 0 & -\frac{i\hbar}{\sqrt{2}} \\ 0 & \frac{i\hbar}{\sqrt{2}} & 0 \end{bmatrix}$$

$$J_z = \begin{bmatrix} \hbar & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -\hbar \end{bmatrix}$$

This will go on for $j = \frac{3}{2}$ (four-dimensional space), $j=2$ (five-dimensional space), etc.

A very important point is values of j that are odd multiples of $\frac{1}{2}$ ($j = \frac{1}{2}, \frac{3}{2}, \dots$). Starting with the orbital angular momentum operator $\vec{L} = \vec{R} \times \vec{P}$ in two or three dimensions, we only find values of m that are integers. This is imposed by the fact that under rotation by 2π the same wave function must be obtained.

$[J_x, J_y] = i\hbar J_z \quad [J_y, J_z] = i\hbar J_x \quad [J_z, J_x] = i\hbar J_y$

However, using the angular momentum algebra (actually the SU(2) algebra, SU(2) is the group of special unitary transformations that preserve the length of a vector in Complex space) we find also values that we cannot find by using the orbital angular momentum operator.

This implies that the angular momentum is more than just the orbital angular momentum.

In fact, the odd multiples of $\frac{1}{2}$ are associated with the so called spin degree of freedom. This is a purely quantum mechanical degree of freedom. The spin angular momentum can assume odd multiples of $\frac{1}{2}$ if the particle is a fermion.

Another important point is that all higher dimensional $(j \geq 1)$ matrix representations of J_x, J_y, J_z can be constructed from the two-dimensional representation $(j = \frac{1}{2})$. This is because the $SU(2)$ group has only one fundamental representation, namely the two-dimensional one. How this is done in practice is related to the addition of angular momenta, which we will not have to discuss in detail (for an example, see your homework assignment #10)